CONCURSUL NAȚIONAL DE MATEMATICĂ
„TEHNICI MATEMATICE”-editia a XIX-a
Etapa judeteană 23.02.2024
Clasa a X -a Matematică *M_șt-nat*

Barem de corectare

Subiectul I (30p)

a) $\begin{cases} 2x+1\ge 0\,;3x+2\ge 0\,;4x+3\ge 0 \\ 1-2x\ge 0\,;2-3x\ge 0\,;3-4x\ge 0 \end{cases}$ 1p

$<=> \begin{cases} x\in\left[-\frac{1}{2},\infty\right)\ ;\ x\in\left[-\frac{2}{3},\infty\right)\ ;\ x\in\left[-\frac{3}{4},\infty\right) \\ x\in\left(-\infty,\frac{1}{2}\right]\ ;\ x\in\left(-\infty,\frac{2}{3}\right]\ ;\ x\in\left(-\infty,\frac{3}{4}\right] \end{cases}$ 1p

$D = x\in\left[-\frac{1}{2},\infty\right)\cap\left(-\infty,\frac{1}{2}\right] => x\in\left[-\frac{1}{2},\frac{1}{2}\right]$....................1p

$\left(\sqrt{2x+1}-\sqrt{1-2x}\right)+\left(\sqrt{3x+2}-\sqrt{2-3x}\right)+\left(\sqrt{4x+3}-\sqrt{3-4x}\right)=0$ 1p

$\frac{2x+1-1+2x}{\sqrt{2x+1}+\sqrt{1-2x}}+\frac{3x+2-2+3x}{\sqrt{3x+2}+\sqrt{2-3x}}+\frac{4x+3-3+4x}{\sqrt{4x+3}+\sqrt{3-4x}}=0$....................2p

$<=> \frac{4x}{\sqrt{2x+1}+\sqrt{1-2x}}+\frac{6x}{\sqrt{3x+2}+\sqrt{2-3x}}+\frac{8x}{\sqrt{4x+3}+\sqrt{3-4x}}=0$....................2p

$x\left(\frac{4}{\sqrt{2x+1}+\sqrt{1-2x}}+\frac{6}{\sqrt{3x+2}+\sqrt{2-3x}}+\frac{8}{\sqrt{4x+3}+\sqrt{3-4x}}\right)=0$....................1p

$<=> x=0\ \in\left[-\frac{1}{2},\frac{1}{2}\right]$....................1p

b) $D = x\in[-2024\,,2024]\setminus\{0\}$ 1p

$\frac{1}{x}\cdot\sqrt{2024^2-x^2}-\frac{1}{|x|}\cdot\sqrt{2024^2-x^2}=0 <=> \sqrt{2024^2-x^2}\cdot\left(\frac{1}{x}-\frac{1}{|x|}\right)=0$ 4p

I) $2024^2-x^2=0 <=> x=\pm 2024$ 2p

II) $\frac{1}{x}=\frac{1}{|x|} <=> x\in(0\,,2024]$ 2p

$S=(0\,,2024]\cup\{-2024\}$ 1p

c) Fie: $\sqrt[3]{\sqrt{5}+2}=a$; $\sqrt[3]{\sqrt{5}-2}=b => x=a-b <=> x^3=a^3-3ab(a-b)-b^3$ ...3p

$<=> x^3=\sqrt{5}+2-3\cdot\sqrt[3]{\left(\sqrt{5}+2\right)\cdot\left(\sqrt{5}-2\right)}\cdot x-\sqrt{5}+2 <=> x^3=4-3x$ 4p

$<=> x^3+3x-4=0 <=> (x-1)\cdot(x^2+x+4)=0 <=> x=1$ 3p

Subiectul II (30p)

a) Notăm $2^x+2^{-x}=t\ (t>0)\ \uparrow^2 => 4^x+2+4^{-x}=t^2 <=> 4^x+4^{-x}=t^2-2$ 2p

$8(t^2-2)-54t+101=0 <=> 8t^2-54t+85=0\ <=>\ t_1=\frac{17}{4}\,;t_2=\frac{5}{2}$ 2p

I) $2^x+2^{-x}=\frac{17}{4}$. Notăm $2^x=a\ (a>0) => a+\frac{1}{a}=\frac{17}{4} <=> 4a^2-17a+4=0$.......1p

$a_1=4\,;a_2=\frac{1}{4}$

Dacă $a_1=4 => x_1=2$ 1p

Dacă $a_2=\frac{1}{4} => x_2=-2$ 1p

II) $2^x+2^{-x}=\frac{5}{2}$. Notăm $2^x=b\ (b>0) => b+\frac{1}{b}=\frac{5}{2} <=> 2b^2-5b+2=0$ 1p

$b_1 = 2\,; b_2 = \dfrac{1}{2}$

Dacă $b_1 = 2 \Rightarrow\ x_3 = 1$ 1p

Dacă $b_2 = \frac{1}{2} \Rightarrow\ x_4 = -1$ 1p

b) $\log_a(a^2b^3) = 1 \Rightarrow\ \log_a a^2 + \log_a b^3 = 1 \Rightarrow 2 + 3\cdot\log_a b = 1 \Rightarrow \log_a b = -\frac{1}{3}$ (1)

$\log_{a^2b^3}\left(\frac{\sqrt[5]{a^3b^2}}{ab^3}\right) = \frac{\log_a\left(\frac{\sqrt[5]{a^3b^2}}{ab^3}\right)}{\log_a(a^2b^3)} = \log_a\left(\frac{\sqrt[5]{a^3b^2}}{ab^3}\right) = \frac{1}{5}\cdot\log_a(a^3b^2) - \log_a(ab^3) =$................4p

$= \frac{1}{5}\cdot(\log_a a^3 + \log_a b^2) - (\log_a a + \log_a b^3) = \frac{1}{5}\cdot(3 + 2\log_a b) - (1 + 3\log_a b)$.........4p

$\overset{(1)}{=}\ \frac{1}{5}\cdot\left(3 - \frac{2}{3}\right) - \left[1 + 3\cdot\left(-\frac{1}{3}\right)\right] = \frac{1}{5}\cdot\frac{9-2}{3} = \frac{7}{15}$ 2p

c) Cum $x \geq e\,; y \geq e\,; z \geq e \Rightarrow \ln x \geq 1\,; \ln y \geq 1\,; \ln z \geq 1 \Rightarrow (\ln x - 1)(\ln y - 1) \geq 0$

$\Leftrightarrow \ln x\cdot\ln y - \ln x - \ln y + 1 \geq 0 \mid \cdot \ln z$ 2p

$\Leftrightarrow \ln x\cdot\ln y\cdot\ln z - \ln x\cdot\ln z - \ln y\cdot\ln z + \ln z \geq 0$ (1) 1p

Analog : $\ln x\cdot\ln y\cdot\ln z - \ln y\cdot\ln x - \ln z\cdot\ln x + \ln x \geq 0$ (2) 1p

$\ln x\cdot\ln y\cdot\ln z - \ln z\cdot\ln y - \ln x\cdot\ln y + \ln y \geq 0$ (3) 1p

Adunăm (1),(2),(3)

$\Rightarrow 3\cdot\ln x\cdot\ln y\cdot\ln z - 2(\ln x\cdot\ln y + \ln x\cdot\ln z + \ln y\cdot\ln z) + \ln x + \ln y + \ln z \geq 0$.................2p

$\Leftrightarrow \ln x + \ln y + \ln z + 3\cdot\ln x\cdot\ln y\cdot\ln z \geq 2(\ln x\cdot\ln y + \ln x\cdot\ln z + \ln y\cdot\ln z)$.........2p

$\Leftrightarrow \ln(xyz) + 3\cdot\ln x\cdot\ln y\cdot\ln z \geq 2(\ln x\cdot\ln y + \ln x\cdot\ln z + \ln y\cdot\ln z)$ 1p

Subiectul III (30p)

a) f – injectivă: $(\forall)x_1, x_2 \in \mathbb{R}\setminus\{1\}\,;\ x_1 \neq x_2$ avem $f(x_1) \neq f(x_2)$ 1p

$f(x_1) = f(x_2) \Leftrightarrow \frac{2x_1-3}{x_1-1} = \frac{2x_2-3}{x_2-1} \Leftrightarrow (2x_1 - 3)\cdot(x_2 - 1) = (2x_2 - 3)\cdot(x_1 - 1)$ 2p

$\Leftrightarrow 2x_1x_2 - 2x_1 - 3x_2 + 3 = 2x_1x_2 - 2x_2 - 3x_1 + 3 \Leftrightarrow\ x_1 = x_2$ (1) 1p

f – surjectivă: $(\forall)y \in \mathbb{R}\setminus\{2\}\,; (\exists)x \in \mathbb{R}\setminus\{1\}$ astfel încât $f(x) = y$ 1p

$\frac{2x-3}{x-1} = y \Leftrightarrow 2x - 3 = xy - y \Leftrightarrow x = \frac{y-3}{y-2} \neq 1 \Leftrightarrow y - 3 \neq y - 2\ (A)$ (2) 2p

Din (1) și (2) => f – bijectivă => f – inversabilă...1p

$f^{-1}: \mathbb{R}\setminus\{2\} \to \mathbb{R}\setminus\{1\}\,; f^{-1}(x) = \frac{x-3}{x-2}$ 1p

$f(x)\cdot f^{-1}(x) = 3 \Leftrightarrow \dfrac{2x-3}{x-1}\cdot\dfrac{x-3}{x-2} = 3 \Leftrightarrow 3(x-1)\cdot(x-2) = (2x-3)\cdot(x-3)$

$\Leftrightarrow 3x^2 - 9x + 6 = 2x^2 - 9x + 9 \Leftrightarrow x^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$.......................................1p

b) $\left(\left|\frac{z_1+z_2}{2} + z\right| + \left|\frac{z_1+z_2}{2} - z\right|\right)^2 = \left|\frac{z_1+z_2}{2} + z\right|^2 + \left|\frac{z_1+z_2}{2} - z\right|^2 + 2\cdot\left|\frac{z_1+z_2}{2} + z\right|\cdot\left|\frac{z_1+z_2}{2} - z\right| =$

$= \left(\frac{z_1+z_2}{2} + z\right)\cdot\left(\frac{\overline{z_1}+\overline{z_2}}{2} + \bar{z}\right) + \left(\frac{z_1+z_2}{2} - z\right)\cdot\left(\frac{\overline{z_1}+\overline{z_2}}{2} - \bar{z}\right) + 2\cdot\left|\left(\frac{z_1+z_2}{2}\right)^2 - z^2\right| =$ 4p

$= 2\cdot\left|\frac{z_1+z_2}{2}\right|^2 + 2\cdot|z|^2 + 2\cdot\left|\frac{z_1-z_2}{2}\right|^2 = |z_1|^2 + 2\cdot|z_1z_2| + |z_2|^2 = (|z_1| + |z_2|)^2$ 4p

$\Leftrightarrow \left|\frac{z_1+z_2}{2} + z\right| + \left|\frac{z_1+z_2}{2} - z\right| = |z_1| + |z_2|$ 2p

c) $z^{2024} = 1 \Rightarrow S = \frac{1}{1+\frac{1}{z^{2023}}} + \frac{1}{1+\frac{1}{z^{2022}}} + \cdots + \frac{1}{1+\frac{1}{z^2}} + \frac{1}{1+\frac{1}{z}} + \frac{1}{2}\ \mid \cdot 2$ 3p

Din $\frac{1}{1+z^{2024}} = \frac{1}{2} \Rightarrow 2\cdot S = 1 + \sum_{k=1}^{2023}\left(\frac{1}{1+z^k} + \frac{1}{1+\frac{1}{z^k}}\right) = 1 + \sum_{k=1}^{2023}\frac{z^k+1}{z^k+1} = 2024$ 5p

$\Rightarrow S = \frac{2024}{2} = 1012$ 2p